Applied Topology Notes

Edward G. Hirst
City, University of London

November 5, 2019

An introduction to the concepts in Applied Topology, adapted from the 2019 LTCC course of the same name given by Dr. P. Skraba. Delivered in the City, University of London, Mathematics PhD Seminars.

1 Simplicial Complexes

Initially we start by introducing the *k-simplex*. This is a generalisation of the triangle, to k dimensions. It is the convex combination of $(k+1)$ points which are affinely independent. Subsimplices of dimension k' of a k-simplex are called k'-faces of the simplex. The simplices may be labelled with an orientation, however in this consideration they are unoriented, and the work is completed over $\mathbb{Z}_2$ which has the same effect as orientation. The first 4 simplices are shown in 1.

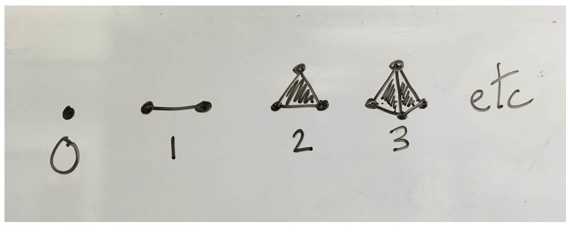

Figure 1: Simplices of dimension 0 to 3.

A simplicial complex, Δ, is a set of simplices, $\{\sigma_a\}$, such that:

$$\begin{aligned} &1) \quad \sigma_a \in \Delta \text{ and } \sigma_b \subseteq \sigma_a \implies \sigma_b \in \Delta\,, \\ &2) \quad \sigma_a \cap \sigma_b \neq \emptyset \implies \sigma_a \cap \sigma_b \in \Delta\,, \end{aligned} \tag{1.1}$$

The dimension of the simplicial complex is the dimension of its largest simplex. Operators on the complex can then be defined. Important ones are:

$\sim$ p-section - the set of all p-simplices in the complex.
$\sim$ Closure ($Cl(\Delta)$) - the set of all simplices in the complex, and all their respective faces.
$\sim$ Boundary ($\partial_p(\Delta)$) - the $(p-1)$-section of the closure of a complex with dimension p.
$\sim$ Star ($St(\sigma)$) - an operator on a simplex in the complex, it is the set of all simplices which contain the simplex in question as a face.
$\sim$ Link ($Lk(\sigma)$) - an operator on a simplex in the complex, it is defined $Cl(St(\Delta)) - St(Cl(\Delta))$.

Examples of the first three operators are shown in figure 2.

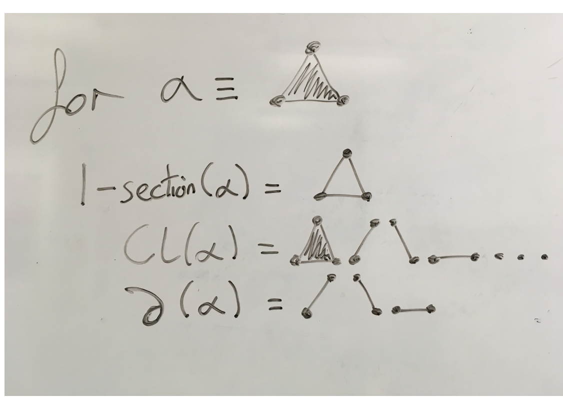

Figure 2: For simplicial complex α, the 1-section, closure, and boundary are given.

As an aside, a useful generalisation of the simplicial complex, is the cellular complex. Originally named the CW-complex, standing for c̲losed, and w̲eakly topological. Instead of the complex being a composition of simplices, it is a composition of cells, where a k-cell is homeomorphic to the k-dimensional open ball. These object and other complexes also find various use in applied topology, however here the focus is on simplicial complexes.

2 Chain Complexes and Homology

For a prespecified simplicial complex, a chain can be defined as a linear combination of simplices from the complex. Therefore $c \equiv \Sigma_a \lambda_a \sigma_a$, where the sum is over the full set of simplices in the complex, and $\lambda_a \in \{0, 1\}$, such that all simplices from the complex are either in, or not in, the chain.

This defines a chain group, C_k, where the elements are all possible chains that can be formed exclusively from k-simplices in the complex. This group's representation is a diagonal matrix of dimension equal to the number of k-simplices in the complex, then the diagonal elements are the λ_a values as described in each chain in question.

Clearly from here, the section, closure, and boundary operators map to chains. Importantly the boundary operator, since it is linear, can act directly on a chain. The boundary operator will map from the C_k chain group to the C_{k-1} chain group. This hence forms a sequence, known as the chain complex: $\{C_k, \partial_k\}$, of all chain groups that can be formed from the complex, and the corresponding boundary operators that map between them. Note that the ∂_0 boundary operator will map to the trivial group as we define $\partial_0(\text{any 0-simplex}) = 0$.

The chain complexes require $\partial_k \cdot \partial_{k+1} = 0$, such that $image(\partial_{k+1}) \subseteq kernel(\partial_k)$. This identifies elements in the kernel of a boundary operator with closed elements, the kernel is known as the 'cycle space', as its elements are cycles, and action of the ∂_k boundary element on a cycle leads to repetition of its $(k-1)$-faces which is zero under the modulo $\mathbb{Z}_2$. Elements in the image of a boundary operator are identified with exact elements, the image is the 'boundary space' as it is mapped to by the boundary operator. All boundaries are trivially cycles, however not all cycles in a chain act as boundaries.

Therefore homology groups are defined for each chain group in the chain complex, which measure the extent to which the sequence fails to be exact. They are defined:

$$H_k \equiv \frac{kernel(\partial_k)}{image(\partial_{k+1})}, \tag{2.1}$$

where the rank of each homology group is known as the Betti number, and is a measure of the number of group generators such that

$$\beta_k \equiv rank(H_k). \tag{2.2}$$

The homology classes thus represent the cycles which are not boundaries, which can be identified with holes in the underlying simplicial complex. Chain complexes can be represented diagrammatically as shown in figure 3.

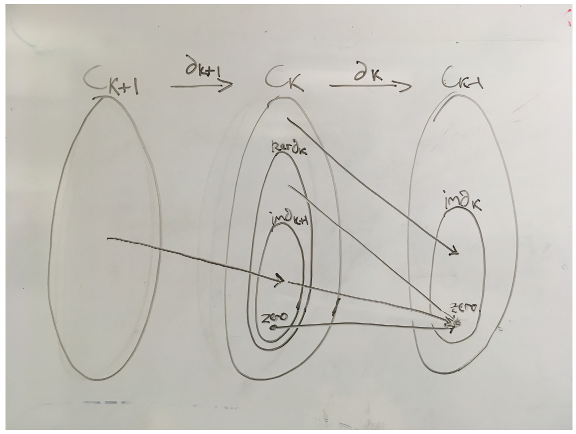

Figure 3: Three chain groups in a chain complex, showing how the boundary operators map between them. The image of ∂_{k+1} is contained within the kernel of ∂_k, if all elements in the kernel are contained within the previous operator's image the homology group is trivial.

To see the practicality of homology groups we examine perhaps the most famous topological invariant. The Euler characteristic is typically defined

$$\chi \equiv \sum_{i=0}^{d} (-1)^i |K_i| \,, \tag{2.3}$$

where d is the dimension of the simplicial complex, and $|K_i|$ is the number of i-simplices in the simplicial complex (i.e. cardinaltiy of the set of i-simplices, K_i). This is easily shown for simple polyhedra by the invariant value $\chi' = (\#vertices) - (\#edges) + (\#faces)$. Which are the 0 to 2 simplices in an alternating sum as generalised above. Therefore noting that the number of i-simplices is the rank of the ith chain group we can write

$$
\begin{aligned}
\chi &= \sum_{i=0}^{d} (-1)^i |K_i| \,, \\
&= \sum_{i=0}^{d} (-1)^i rk(C_i) \,, \\
&= \sum_{i=0}^{d} (-1)^i (rk(ker(\partial_i)) + rk(im(\partial_i)) \,, \\
&= \sum_{i=0}^{d} \Big((-1)^i (rk(ker(\partial_i)) - rk(im(\partial_{i+1})) \Big) + (rk(im(\partial_0)) - rk(im(\partial_d))) \,, \\
&= \sum_{i=0}^{d} (-1)^i (rk(ker(\partial_i)) - rk(im(\partial_{i+1})) \,, \\
&= \sum_{i=0}^{d} (-1)^i rk(H_k) \,, \\
&= \sum_{i=0}^{d} (-1)^i \beta_k \,.
\end{aligned} \tag{2.4}
$$

In the above, the chain group is split into kernel and image of boundary operator, as it is a monomorphism such that everything is mapped. The second sum is shifted by one index, and the images of the boundary operators either end of the sequence are written explicitly, although both are defined to be zero in the sequence so are subsequently ignored. The result is the Euler characteristic as the alternating sum of Betti numbers, showing the relevance of homology classes in determining this invariant.

As another aside, a cochain complex is the category theory dual of a chain complex. Its coboundary operators d^k map between the dual chain groups in the direction of increasing simplex dimension such that $d^k : C^{k-1} \mapsto C^k$. The coboundary operators satisfy $d^k \cdot d^{k-1} = 0$; and the cohomology groups measure the failure of exactness of the cochain complex with

$$
H^k \equiv \frac{kernel(d^k)}{image(d^{k-1})} \,. \tag{2.5}
$$

In the field of differential geometry the de Rham cochain complex studies forms in the cochain groups, with the exterior derivatives as the coboundary operators.

3 Chain maps and Persistent Homology

Firstly we define a filtration, which is a series of simplicial complexes such that

$$\Delta_a \supseteq \Delta_b \supseteq \Delta_c \supseteq \ldots \supseteq \Delta_n \,. \tag{3.1}$$

A practical way to think about this is that each simplicial complex represents a triangulation of the single underlying topological space, however the complexes which are subcomplexes of others are triangulations at lower resolutions. In the same way as before for each simplicial complex in the filtration we can create a chain complex. The chain groups between complexes are then connected by monomorphisms, and the full set forms a chain map known as a persistent chain complex. This chain map is a commutative diagram between boundary operators and monomorphisms along the filtration. Part of a chain map is shown in figure 4.

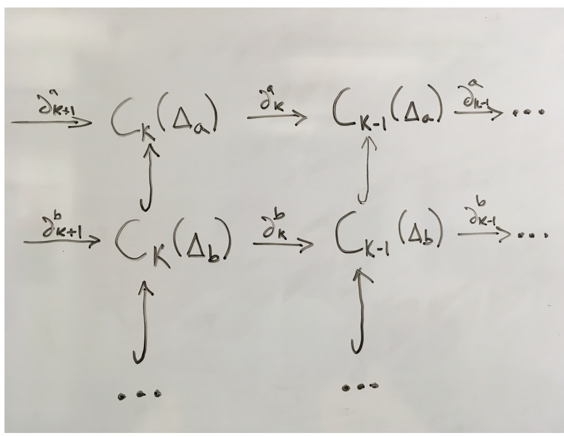

Figure 4: A chain map, where each row is a chain complex for a different simplicial complex in the filtration. Columns are the monomorphisms along the filtration for each chain group dimension. Overall this produces a commutative diagram.

Constructing homology groups, as before, for each chain group in the chain map allows a persistence module to be defined for each simplex dimension k, where the monomorphisms between the chain complexes now correspond to linear maps. Therefore

$$P_k(filtration) \equiv \{H_k(\Delta_i)\}_{\forall i} \;\&\; H_k(\Delta_i) \mapsto H_k(\Delta_j) \,. \tag{3.2}$$

Moving down the filtration homology classes are born, where holes are made by removing simplices whilst leaving their cyclic boundaries. In addition classes also die, where simplices are removed from a cycle bounding a hole (connecting the hole to the simplicial complex boundary), or the simplices connecting two holes are removed merging the holes and hence classes.

Due to Gabriel's theorem for finite quivers, the persistence modules can be represented by this birth/death class structure as a 'barcode'. The interpretation of this barcode decomposition is still open. It can be represented diagrammatically using a lower-star filtration function $f : \Delta \mapsto \mathbb{R}$ which is monotonic such that each complex $\Delta_i \sim f^{-1}(-\infty, i]$ is contained as a sublevel set within the function values for the larger complexes in the filtration.

Plotting the homology classes throughout the filtration, and using the Elder rule which states the younger homology class dies when classes merge, the barcodes become clear, as shown in figure 5.

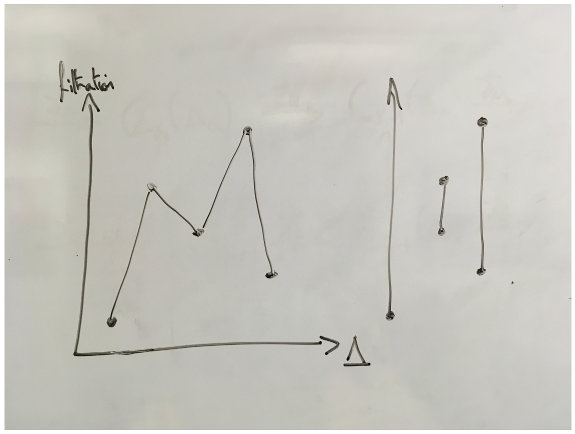

Figure 5: A plot of a lower star filtration to represent the simplicial complex along the horizontal axis, and motion through the filtration along the vertical axis. Moving along the filtration new homology classes are born as components and die as the components merge. This gives the barcode on the right.

The barcodes can then be used to define persistence diagrams for the classes, as shown below. From here multiple persistence diagrams can be compared with bottleneck distance to determine the stability of the filtration representation of the underlying topological space.

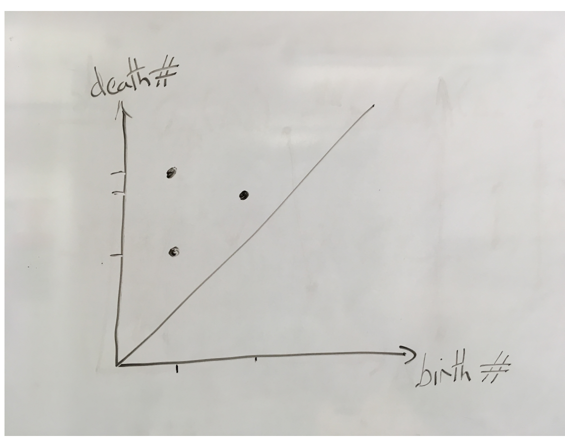

Figure 6: A simple example of a persistence diagram. Here two classes are born at the same point in the filtration, however one dies later, and another is born later but dies earlier. These diagrams are useful in determining stability of the filtration.

References

[1] http://www.ltcc.ac.uk/timetable/

[2] Elementary Applied Topology, Robert Ghrist, 2014

[3] Computational Topology: An Introduction, Edelsbrunner, Harer, 2010