November 4, 2019

Contents

Abstract

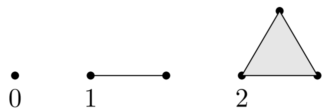
These notes are typed up from the friday sessions at city-university of london. The presentation was delivered by Ed Hirst from a course on topology given by the university. this is a work in progress and the notes have been changed enough that remnant mistakes and typos are most certainly mine: Please send corrections to gabriel.bliard@city.ac.uk or gabriel.bliard@physik.hu-berlin.de if you spot any.

1 Intro to topology

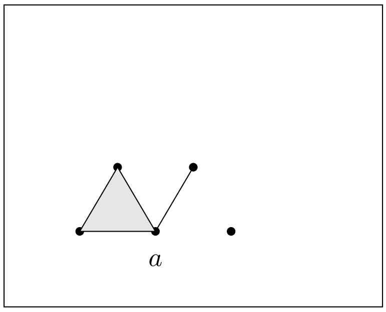
1.1 Objects and operators

Definition 1.1. Simplex

A simplex is a simple extension to the notion of triangle in d-dimensions. It can be defined by a list of points which are the vertices of the simplices. One can introduce an ordering to the list which naturally classifies the simplex in a $\mathbb{Z}_2$ classification (even/odd permutations-sign of the volume element in the language of forms).



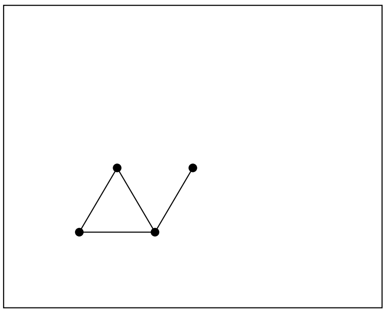
We can then form sets of simplices $\Delta = \{\sigma\}$ which will have a highest order simplex. Before we begin a long list of definitions and theorems, it will be sueful to keep an example in mind to demonstrate these:

$\Delta =$ 

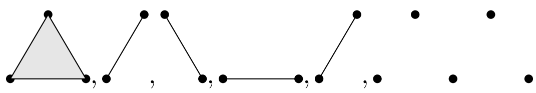
Theorem. *Let Δ be a set of simplices.*
The highest order simplex implies the existence of all lower order simplices.
The intersection of simplicies are again simplicies

1. $\sigma_1 \in \Delta, \tau \leq \sigma \Rightarrow \tau \in \Delta$
2. $\sigma_1 \cap \sigma_2 \neq \emptyset, \quad \sigma \cap \sigma_2 \in \Delta$

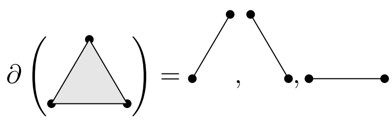
Definition 1.2. A p-section: all simplices in the simplex system of order p



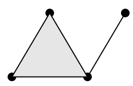
Definition 1.3. Closure Is the set of all p-sections in the simplex system



Definition 1.4. Boundary The boundary of a simplex system of order p is its p-1 section.

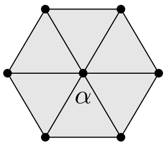


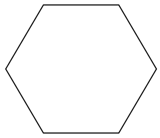
Definition 1.5. Star This gives the faces that contain a given simplex:

$St(a) =$ 

Definition 1.6. Link

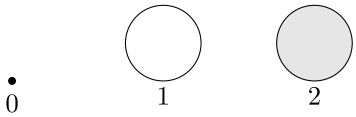
$$\mathrm{Lk}(\alpha) = [\mathrm{Closure}, \mathrm{Star}] \tag{1.1}$$



$\mathrm{Lk}(\alpha) =$ 

One can also define a different set of elements can cellular complex

Definition 1.7. Each element in the set of cellular complexes is homeomorphic to a d-dimensional ball. Note that each cellular complex can be mapped to a set of simplices



Definition 1.8. A chain is a linear sum of Simplices:

$$C = \sum_a \lambda_a \sigma_a \tag{1.2}$$

Note that the boundary of the a chain is well defined because of the linearity of the bdry operator
However, simplices are not the only object that can be considered in this framework. Cellular complexes provide a similar arena for topology. I paticular, one can pave n-cellullar complexes which n-simplices.

1.2 Chains and maps

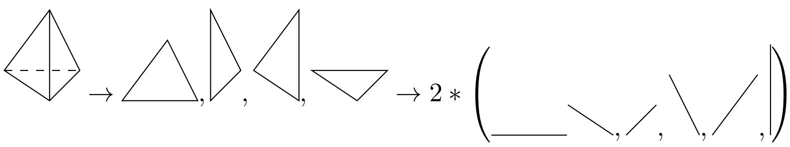
The main object in topology will the chain

Definition 1.9. Chain Linear sum of simplices:

$$C = \sum_a \lambda_a \sigma_a \tag{1.3}$$

Note that the boundary operator acts naturally on chains to give an n-1 chain.

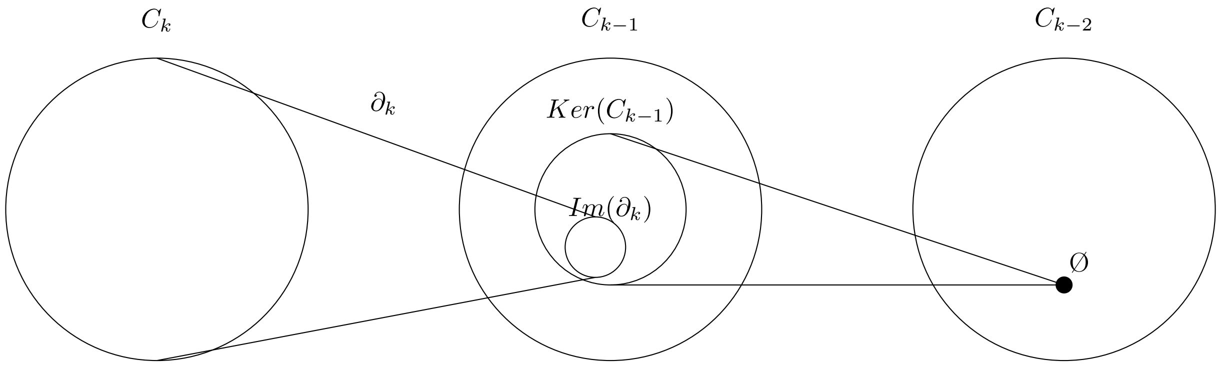
The successive application of two boundary operators is nilpotent. This is trivial for $n \preceq 1$ since the boundary operator annihilates point (or 0-simplices). /par However, for higher dimensional simplices, one must invoke the notion of ordering, orientation, or $\mathbb{Z}_2$ modularity. Under these conditions, it is easy to see that tetrahedra as well as higher order simplices will vanish upon operation of two successive boundary operators.



Where this was reasoned in terms of $\mathbb{Z}_2$ modularity, but the same result is even more intuitive when considering orientable surface (Just add arrows to the above diagrams).

If we call C_k the set of all chains of highest simplex k, we can see that the boundary operator maps these into one another:

$$C_k \rightarrow^{\partial_k} C_{k-1} \rightarrow^{\partial_{k-1}} C_{k_2} \rightarrow ... \rightarrow 0 \tag{1.4}$$



From this we have

$$\mathrm{Im}(\partial_{k+1}) \in \mathrm{Ker}(\partial_k) \tag{1.5}$$

If the case of equality the sequence is called exact, and the homology group is then trivial.

Definition 1.10. Homology group

$$H_k := \frac{\mathrm{Ker}\partial_k}{\mathrm{Im}\partial_{k+1}} \tag{1.6}$$

From this we find the β-function of the map : $\beta_k = r_k(H_k)$

When one has multiple chains, we can define chain maps between them. Let

$$\delta_1 = \{C_k, \partial_k\} \tag{1.7}$$

$$\delta_2 = \{B_k, \partial_k\} \tag{1.8}$$

$$\tag{1.9}$$

We can then define a filtration if $\delta_1 \in \delta_2 \in \delta_3 .. \in \delta_n$ which in turn is the coarse or fine-graining of the simplectic system.

1.3 Examples

Consider the triangulation of S^2 by 2-simplices. this is just a full triangle in a full disc. we represent this by 3 point (a,b,c) in the plane. The 2-simplex abc then map onto ab ac and bc through the matrix :

∂_2	abc
ab	1
ac	1
bc	1

Likewise, the simplices thus obtained map onto the points abc through the matrix:

∂_2	ab	bc	ca
a	1	0	1
b	1	1	0
c	0	1	1

The successive application of these maps (eq to the matrix multiplication)leads to a trivial map $\partial_2 \circ \partial_1 = 0 \mod(2)$

References