Chern number: oriented manifold of dim $2n$,
integrate the product of Chern classes with the sum of indices being n

→ get an integer called Chern number

→ Hence, # of Chern numbers is # of partitions of n

e.g. $\dim_{\mathbb{C}}(M) = 3$ (i.e. $\dim_{\mathbb{R}}(M) = 6$)

$$C_1 = \int_M c_1^3(M), \quad C_2 = \int_M c_1(M)\, c_2(M), \quad C_3 = \int_M c_3(M)$$

Note that $\int_M c_n(M) = \chi$ → top Chern number is Euler characteristic

For first Chern number:

M, N : $2n$-dim'l compact oriented manifold

cts $f : M \to N$

Let $\pi : E \to N$ be a fibre bundle with fibre F (can replace M & N with any topological base spaces B' & B in pullback bundle), define pullback bundle:

$$f^*E = \{(p,e) \in M \times E \mid f(p) = \pi(e)\} \subseteq M \times E$$

and equip it with subspace topo & projection $\pi' : f^*E \to M$ by

$$\pi'(p, e) = p$$

and with another projection $h : f^*E \to E$,

we have the following commuting diagram:

$$\begin{array}{ccc} f^*E & \xrightarrow{h} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{f} & N \end{array}$$

f^*E is called the pullback of E by f, or the bundle induced by f.

Prop. Naturality: Chern classes: $c(f^*E) = f^*c(E)$

" forms : $c_k(f^*E) = f^*c_k(E)$

def. The degree of f: $c_1(f^*E) = \deg(f) \cdot c_1(E)$

alternative def. $f: M \to N$ induces a homomorphism

$f_*: H_n(M) \to H_n(N)$ on homology gps

$[X] \mapsto [Y]$

Then $f_*([X]) = \deg(f)[Y]$

In fact, f induces two homs:

$H_n(f) = f_*: \quad \mathbb{Z} \cong H_n(M) \to H_n(N) \cong \mathbb{Z}$, $[x] \mapsto w[x]$

& $H^n(f): \quad \mathbb{Z} \cong H^n(M) \to H^n(N) \cong \mathbb{Z}$, $[x] \mapsto w[x]$

Then $\deg(f) \equiv w$

$\longrightarrow$ $\deg(f)$ is also known as the winding number of f

(perhaps more obvious with homotopy & based loops)

Then the Chern numbers of M & N are related by:

$$\int_M c_1^{a_1} c_2^{a_2} \cdots c_k^{a_k} = (\deg(f))^n \int_N c_1'^{a_1} \cdots c_k'^{a_k} \qquad \left(\sum_{i=1}^{k} a_i i = n\right)$$

by naturality

e.g. $\int_M c_1 = \deg(f) \int_N c_1'$

esp. first Chern number $C_1 = \int_M c_1(E) = \int_M ch_1(E)$

e.g. Consider $U(1)$ bundle over S^2 & $A = \frac{1}{2}in(1-\cos\theta)d\varphi$
(omit $\hbar$ & e ---)

$$\Rightarrow F = dA + \cancel{A\wedge A}^{\,0} = -\frac{1}{2}in\sin\theta\, d\varphi\wedge d\theta$$

$$\Rightarrow ch_1(F) = \frac{i}{2\pi}F = \frac{n}{4\pi}\sin\theta\, d\varphi\wedge d\theta$$

$$\Rightarrow 1^{st} \text{ Chern number}: \int_{S^2} ch_1(F) = \frac{n}{4\pi}\int_{S^2}\sin^2\theta\, d\varphi d\theta$$

$$= n$$

topological charge of Dirac monopole

e.g. (Berry phase) $G = U(1)$, $M = S^2$

(Berry) connection : $A^{(+)} = -\sin^2\frac{\theta}{2}d\varphi$, $A^{(-)} = -\cos^2\frac{\theta}{2}d\varphi$

$\Rightarrow$ (Berry) curvature: $F^{(\pm)} = \pm\frac{1}{2}\sin\theta\, d\varphi\wedge d\theta$

$$\Rightarrow c_1^{(\pm)} = \left[\frac{i}{2\pi}\mathrm{Tr}\,F^{(\pm)}\right] = \pm\frac{i}{4\pi}\sin\theta\, d\varphi\wedge d\theta$$

$$\Rightarrow C_1^{(\pm)} = \int_{S^2} c_1^{(\pm)} = \pm i$$

Let $f: T^2 \to S^2$ s.t. f^*E is the $U(1)$ bundle above
(Brillouin zone)

$$\Rightarrow C_1^{(\pm)}(T^2) = \deg(f)\, C_1^{(\pm)}(S^2) = \pm i\deg(f)$$

(in physics, $c_1^{(\pm)}$ is $[\frac{1}{2\pi}F] \Rightarrow C_1^{(\pm)}$ are ± 1 & $\pm\deg(f)$)